Things of science

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LINKAGES

Unit No. 331

LINKAGES

This unit of THINGS of science contains bars of different lengths, eyelet fasteners, washers, a nail, heavy paper platforms and colored stiff paper.

Ordinarily when we observe geometric constructions, such as a square, triangle or circle, we think of them as being fixed in space, or static, each with a definite dimension that does not change. Geometry, however, also makes use of the concept of motion. This field of geometry that is concerned with motion is known as kinematics.

One of the most interesting branches of kinematics is that of "plane linkages," a field related to elementary plane geometry. In this unit we shall construct some of these linkages and observe their behavior.

Plane linkages are made up of rigid rods that are either connected to each other with movable joints or to fixed points around which they can turn freely in either direction.

The construction of a straight line by means of linkages has been a famous problem attempted by many mathematicians before it was finally solved.

This unit includes materials for constructing linkages that provide a means of

producing a straight line as well as other curves. We say "other" curves since the word curve refers to a class of figures of which the straight line is a member.

First identify your specimens.

BARS — Made of strong stiff paper; several sizes; the measurement of the bars is the distance between the centers of the holes in the bars.

EYELETS—Metal eyelets to link the bars together.

WASHERS—To hold eyelets in position; removable and reusable.

NAIL—For making holes for eyelets.

PLATFORMS—Heavy card with punched holes to serve as bases to which the linkages will be pivoted.

COLORED CARDS—For making additional bars.

CIRCULAR MOTION

Describing a circular motion on a plane surface is easy. All one needs is a piece of cardboard or other flat object that will not stretch or compress. If you pivot one point of a piece of cardboard and then rotate the cardboard in a plane about this point, any other point on the cardboard will trace a perfect circle around the fixed point.

This is an example of the simplest

form of linkage and is the principle on which the compass is based.

Experiment 1. Make such a linkage from any handy material. This type of simple linkage is in general use in our daily lives, not only in compasses but in commonplace mechanisms as swinging doors, pivoted lamps and rotary fans. All of these describe circles as they move on a plane. Look around you. Can you see other examples of this type of linkage?

There are two basic requirements for all linkages. The bar must be rigid and its length incompressible so that the distance between any two points will remain constant.

The shape of the bar and its thickness or thinness are not important.

Experiment 2. Cut out a straight bar with points two inches apart using one of your colored papers. Cut out another of irregular shape with points also

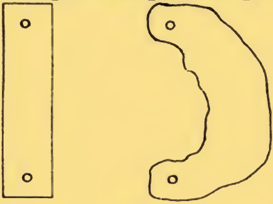

Fig. 1

two inches apart (Fig. 1). Pivot one end of each of these bars with a pin and draw circles. Note that the shape does not interfere with the function of the compass.

For convenience, however, straight bars are used in all the experiments here.

ASSEMBLING THE LINKAGE

To join the bars in making linkages, insert the eyelet in the hole and telescope the washer over the stem of the eyelet from beneath. Use the eyelets and washers for all links and pivots.

When tracing a line, the pencil is inserted into the eyelet. To prevent the pencil from moving within the eyelet, surround the point with a little cotton or tape or other suitable material, or use a stubby pencil that will fill up the hole.

The washer can be easily removed from the eyelet for reuse in the various experiments. If the washer becomes loose just squeeze the inner rim slightly with pliers or forceps.

Take the base card having the three series of holes. Number the first section, the one containing four holes, 1, the middle section 2 and the last one 3. Number the holes in the first section from left to right. The numbering of these

parts is purely for identification. Separate the three sections.

When the bases are used, slip a sheet of plain paper sufficiently large between the base and linkage for your tracings, passing the eyelet through both the paper and base. Use your nail to punch the proper size hole in the paper.

PANTOGRAPH

The common pantograph, an instrument for mechanically copying maps or illustrations according to a desired scale, is a well-known plane linkage.

Experiment 3. Let us construct a pantograph.

A pantograph is based on the geometry of a parallelogram, a geometric figure with opposite sides equal and parallel. It is a parallelogram with two extended legs. Usually the basic parallelogram is a rhombus with all sides equal.

Take six 4-inch bars and join them to form the rhombus PQRS′ as the basic parallelogram (Fig. 2).

Tape bars OP and PQ together and QR and RS together so they will remain rigid, or if you wish, construct two eight-inch bars from thin cardboard and use them instead. Links PS′ and RS′ should remain movable at points P and R.

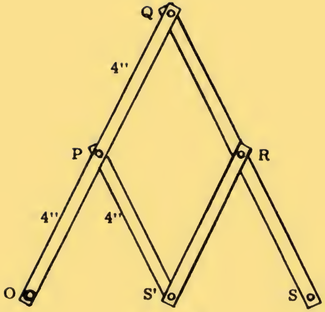

Fig. 2

Fix point O to a hole in one of the base cards. In your structure, $OS' = PR = S'S = \frac{1}{2}(OS)$, since they are always parallel to each other and points O, S′ and S are collinear, or lie along a straight line. Show this is so by measuring the distances with a ruler.

Insert a pencil at point S and another one at S′. Trace a line from S′. How far does S move? Measure the distance with a ruler. Does S move twice the distance that S′ travels?

Move S′ sideways. Does S also move sideways?

Repeat the experiment tracing a line from point S. How far does S′ move?

You have observed the principle of the pantograph.

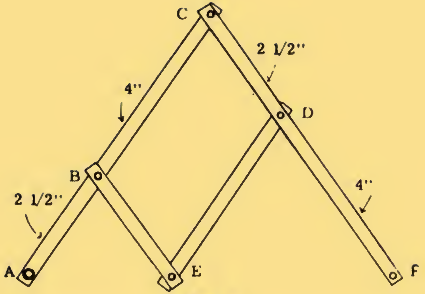

Fig. 3

Experiment 4. The pantograph does not have to be based on a rhombus. Demonstrate this by constructing a pantograph with three 2½-inch bars and three 4-inch bars (Fig. 3).

Always be sure in your constructions that the tracing point E and reproducing point F are collinear with the pivot point A. Note BD does not equal EF. Why?

To calculate the enlargement of a figure at F use the ratio AB/AC. Here, AB/AC = 5/13. Move E ⅝ inch. How many eighths of an inch does F move? Note that the ratio of the movement of E to the movement of F is also 5/13. Construct other pantographs.

STRAIGHT LINE

No construction in geometry is prob-

ably so readily accepted as that of constructing a straight line. But constructing a straight line is not so easy as it seems.

How would you construct a straight line?

You know that you cannot very easily represent a straight line by a freehand drawing. So you will probably reach for a ruler or other straight-edged object and trace a line using it as a guide. If the ruler is straight your representation of a line will be straight. But your representation of a line is dependent on the ruler. It is not a geometric line.

We say "represent" a line because the tracing you made on the paper is not a line. A line has no width and extends an indefinite distance in each direction. No drawing can show these properties. Likewise, triangles, squares, rectangles and circles may be represented by drawings, but they will not be triangles, squares, rectangles or circles. Drawings are simply aids to help us visualize lines and figures.

Euclid postulated the straight line in 300 B.C. but it was not until more than 2,000 years later that a mechanical device was invented by which a straight line could be constructed.

Although mathematicians had been us-

ing the straight line for many centuries, it was the pressure of practical mechanics, the invention of the steam engine in particular, that brought about the intensive research for discovering a way to construct a straight line in a plane.

Euclidian geometry is based upon two fundamental theories: First, that it is possible to construct a straight line through any two points, and second, that it is possible to construct a circle with any point as center and any given radius. The latter you have already demonstrated.

Our problem then is the construction of a mechanism that will construct a straight line but will not use a straight line in its creation.

To do this we turn to plane linkages.

WATT'S LINKAGE

James Watt, one of the inventors of the steam engine needed a mechanical device that would guide the piston of the steam engine in a straight line.

As a result of his experimenting, he devised the first known plane linkage in 1782. This linkage consists of three bars with the two free ends pivoted to a base, and is known as Watt's Parallel Motion.

Experiment 5. Let us make a Watt's

linkage to see whether it actually creates a straight line.

To assemble the linkage, take two four-inch bars from your unit and the 2½-inch bar with the center hole.

Link the bars together as shown in Fig. 4.

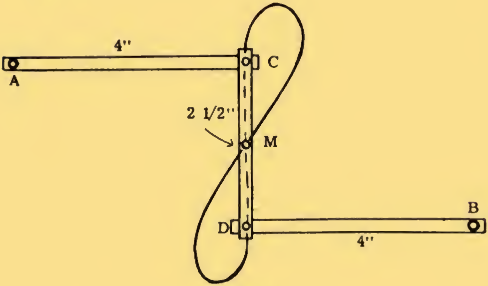

Fig. 4

Obtain heavy paper or thin cardboard large enough to mount the linkage. Check the figure to see which bars should be on top and which underneath in assembling the linkage in this and the following experiments.

Place an eyelet and washer in the center hole of the transverse bar CD also.

With your nail puncture a hole through the paper at points A and B.

Pivot the linkage at points A and B, being sure AC is parallel to DB and the transverse bar CD is perpendicular to both.

Gently move the linkage back and forth to see that the joints are secure and move freely.

Place a pencil in the eyelet at M. Trace a line firmly, but carefully, so that you do not bend the bars, pushing the pencil along, but allowing the linkage to determine the direction. Do not force the movement of your pencil. As you follow the motion limited by the linkage, you will find that a complete circuit will produce a figure eight.

Remove the linkage and you will observe that no part of the figure is perfectly straight. However, close to the center of the figure near the crossing point or node, the lines are nearly straight. Place a ruler or other straight edge along the lines to show this is so.

Because the tracer is halfway between the two pivots and cannot curve either way an approximate straight line results for a short distance. Nowhere is the line absolutely straight and it will curve in one direction or the other after the pencil gets away from the center.

For Watt's purposes, however, the

linkage was accurate enough. He was able to attach the piston of his engine to the midpoint of the linkage and by regulating the length of the engine's stroke the piston could be guided in a nearly straight line.

THE PEAUCELLIER CELL

In 1864, A. Peaucellier, an officer in the French army, discovered the linkage that provided the first exact straight line motion in a plane. This linkage is named after him and is called the Peaucellier Cell.

It consists of four equal links pivoted together to form a rhombus (Fig. 5). To two opposite angles of this rhombus, ABCD, are joined two rods of equal length, OA and OC which are pivoted at O.

The three points OBD lie along a

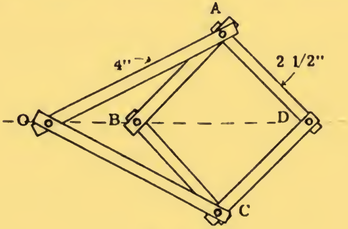

Fig. 5

straight line and are collinear.

Experiment 6. Using your four 2½-inch bars and two 4-inch bars, join them as shown in Figure 5.

Pivot point O in the first hole on the cardboard base No. 1. Change the shape of the figure by increasing or decreasing the angles at points A, B, C or D.

Note that points O, B and D always remain collinear no matter how the structure is deformed.

The most remarkable property of the Peaucellier Cell is that the product of the distances OB and OD is always a constant and will always be the same for a particular cell.

Experiment 7. Measure OB and OD accurately with the linkage in a variety of positions. Multiply the two measurements together for any given position. You will find that the product will always be approximately the same for this particular cell, 9.75.

Experiment 8. Cut out four bars two inches long from one of the colored cards in your unit. Punch holes 1½ inches apart (the distance between them from the center of the holes) with the nail in your unit. Pivot opposite vertices as before to the 4-inch bars.

Is the product of OB and OD constant in this figure also? Are points O, B and D collinear no matter how the structure is deformed?

The product (OB) x (OD) is constant for each Peaucellier Cell.

Experiment 9. The problem now is to make two points move so that the product of their distance from a fixed point is a constant. This is what the Peaucellier Cell does.

How can the cell do this?

From Experiment 7 we know that (OB) x (OD) is a constant (Fig. 6).

Therefore, if point B (Fig. 6), is moved along successive positions on a circle on which the fixed point, point O, is located, then the other point, point D, should travel along a straight line.

Let us see if this is so.

Experiment 10. Take the rod measuring 1¼-inch and attach one end to B. Pivot the other end in the third hole of the base and call it P. Point P, if you will measure the distance, is 1¼-inch from O, a distance equal to the length of bar BP. Thus BP will describe a circle which will pass through O (Fig. 6).

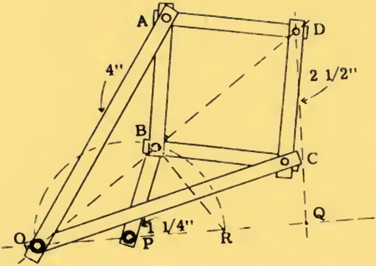

Fig. 6

Place a pencil in the eyelet at D and trace a line down to line OPR. Allow the linkage to determine the direction of your tracing.

Mark the point at which D intersects the base line OPR, Q.

Is DQ a straight line? Is it perpendicular to OPR?

Are points OBD collinear at all times?

Now place your pencil in the eyelet at B and trace a line so that point B describes a circle OBR.

Does D travel along a straight line to Q?

You have constructed a linkage that produces a straight line.

The Peaucellier Cell carries out the geometric transformation known as in-

version. The linkage therefore is a mechanical inversor and can be used to illustrate a number of theorems on inversion.

Experiment 11. Note that there are two similar right triangles in this linkage, OBR and OQD having the same base angle DOQ.

Can you prove from this also that D will always lie on the perpendicular DQ and therefore trace a straight line?

Experiment 12. If the point P is placed so that the distance PO is greater or less than BP, then what happens to the line described by point D?

Place point P in the second hole, ½ inch to the left. Now trace a line from D to Q. What kind of a line do you obtain?

Place P in the hole ½ inch to the right. What type of curve do you produce this time?

From these results, you can see that D will not describe a straight line unless point B can describe a circle which passes through point O.

Can you explain why?

What happens to the product (OB) x (OD) when the position of point P is changed?

Experiment 13. There are a number of other linkages that create a straight line. Here is one that is based on a principle quite different from that used to produce a straight line in the Peaucellier Cell.

Take three 4-inch bars, including the one with the hole along its length, three 2½-inch bars and one 1½-inch bar. Join them as shown in Fig. 4.

Pivot two 4-inch bars at points A and D to the base holes on base card No. 2. Join the 2½-inch bar EF and the 1½-inch bar FG under the other bars at E and G so that point F can move under bar BC.

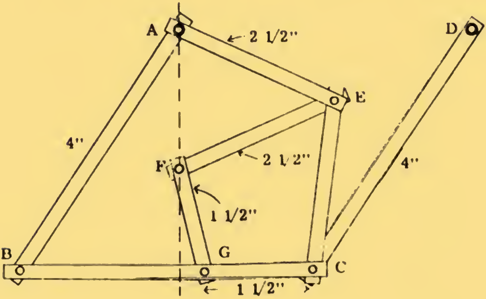

Fig. 7

Place your pencil in the eyelet at F. Trace a line along the path allowed by the structure. Do you get a straight line? In which direction?

Bar FG wants to describe a circle which would pass through C, while bar EF attempts to describe a circle that will pass through A. The result is a straight line.

Note that the outer structure of this linkage system is a rhombus. Observe also that FE = EC = AE and FG = GC. From these observations, can you construct a larger linkage like this that will produce a straight line? Use thin cardboard or other stiff inflexible material for the bars.

THE CISSOID CURVE

There are many other curves of interest in mathematics. One of these curves called the Cissoid of Diocles, can be traced by using a variation of the Peaucellier Cell.

Experiment 14. Join four 2½-inch bars and two 4-inch bars to form a Peaucellier Cell as before, but in this model place the two outer bars of the rhombus, BC and CD, on top of the other bars (Fig. 8). Link a 1½-inch bar at A.

Then pivot point E in one of the holes in base card No. 3. and F in the other

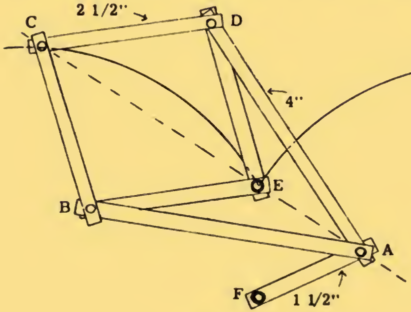

Fig. 8

hole. These holes are 1½ inches apart and equal in length to the bar AF.

Place your pencil in the eyelet at C and move to point E. Keeping your pencil in C, shift point A so that it describes a circle around F until points A, F, E and C are collinear. Then direct your pencil in the opposite direction away from point E and you will achieve a curve which has a sharp point or cusp at E.

TRISECT AN ANGLE

Trisecting an angle with a linkage is very simple.

Experiment 15. Construct a rhombus with the four 2½-inch bars and extend sides AB and AD with 4-inch bars (Fig. 9).

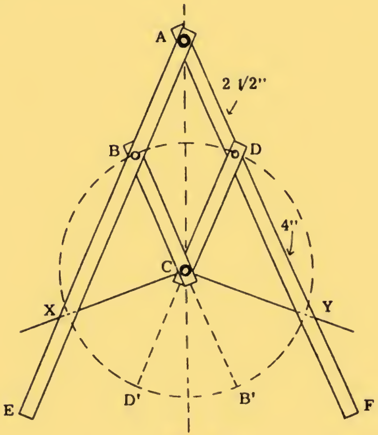

Fig. 9

On a large sheet of paper, draw an angle which you wish to trisect.

Punch a hole with a nail in the vertex of the triangle and secure the rhombus to the paper at C.

Tape bars AB and BE at point B and AD and DF at point D. Bars ABE and ADF should be straight and rigid.

Place a pencil in eyelet B and describe

a circle with radius CB = CD. The circle intercepts the angle at X and Y. Move the extended legs so that they pass through X and Y.

Pivot point A to the paper.

Extend line BC to B′ and DC to D′ BCB′ should be parallel to ADF and DCD′ parallel to ABE.

Arc BD = arc XD′ = arc B′Y, since parallel lines intercept equal arcs on a circle.

Arc BD = arc D′B′ since they are arcs of equal angles, BCD and D′CB′.

Therefore arcs XD′, D′B′ and B′Y are equal and their angles are also equal.

You have trisected the angle XCY by means of a simple plane linkage.

These are some examples of plane linkages and their applications. There are also other types of linkages such as space, polyhedral and spherical linkages.

Linkages can be used for many other purposes, such as tracing any algebraic curve, solving any algebraic equation, finding any root or power of a number and doing operations on complex numbers.

The study of linkages is an interesting branch of mathematics. The theories in linkages have as sound a foundation as the other theories of mathematics. For those who wish to study this subject further, references are listed below.

"A General Method for the Construction of a Mechanical Inversor," M. H. Ahrendt, *The Mathematics Teacher,* 37, 76.

"Linkages", J. Hilsenrath, *The Mathematics Teacher,* 30, 277.

Analysis of the Four Bar Linkage, J. A. Hrones and G. L. Nelson, Wiley, 1951.

Squaring the Circle, E. W. Hobson et al., "How to Draw a Straight Line", A. B. Kempe, Chelsea Publishing Co., 1953.

"Linkages as Visual Aids", B. E. Meserve, *The Mathematics Teacher,* 39, 372.

"Linkages", R. C. Yates, *Eighteenth Yearbook of the National Council of Teachers of Mathematics,* p. 117. Bureau of Publications, Columbia University, New York, 1945.

"The Story of the Parallelogram", *The Mathematics Teacher,* 33, 301.

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Geometrical Tools, R. C. Yates, Educational Publishers, St. Louis, 1949.

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